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Abstract

Random variables $\underset{\sim}{T} = \{T_1,\ldots,T_n\}$ are *associated* if $\mathrm{Cov}[f(\underset{\sim}{T}),g(\underset{\sim}{T})] \geq 0$ for all increasing f,g for which the covariance exists. T_{n+1} is stochastically increasing in $T_1,\ldots,T_n$ if $P[T_{n+1}>t_{n+1}|T_1=t_1,\ldots,T_n=t_n]$ is increasing in $t_1,\ldots,t_n$ for each fixed t_{n+1}. In this paper, results of the following type are derived: If T_i is stochastically increasing in $T_1,\ldots,T_{i-1}$ for $i = 1,\ldots,n$, then $T_1,\ldots,T_n$ are associated. Examples are given of the application of these results to reliability models involving various types of maintenance.

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Generating Associated Random Variables

1. Introduction

A set of random variables $\underset{\sim}{T} = \{T_1,\ldots,T_n\}$ are said to be *associated*, if $\mathrm{Cov}[f(\underset{\sim}{T}),g(\underset{\sim}{T})] \geq 0$ for all increasing functions f,g for which the covariance exists (an increasing function is a function which is nondecreasing in each of its arguments). Esary-Proschan-Walkup (hereafter referred to as E-P-W) (1967) develop the basic properties of associated random variables and present some applications [see also E-P-W (1966) for applications to reliability theory]. Tukey (1958) discusses the notion of *positive regression dependence* of T_2 on T_1, defined by the property that $P[T_2>t_2|T_1=t_1]$ is increasing in t_1 for each fixed t_2. Lehmann (1966) discusses several forms of bivariate dependence, including positive regression dependence (but not bivariate association), shows their relationships, and gives a number of applications. Esary and Proschan (1967) discuss the relationships between bivariate association and the forms of bivariate dependence considered by Lehmann.

It is shown in E-P-W (1967) that positive regression dependence of T_2 on T_1 implies association between T_1 and T_2. In the present paper we define T_{n+1} to be *stochastically increasing* in $T_1,\ldots,T_n$ if $P[T_{n+1}>t_{n+1}|T_1=t_1,\ldots,T_n=t_n]$ is increasing in $t_1,\ldots,t_n$ for each fixed t_{n+1}, and show:

Theorem 1.1. Let $T_1,\dots,T_n$ be associated. Let T_{n+1} be stochastically increasing in $T_1,\dots,T_n$. Then $T_1,\dots,T_{n+1}$ are associated.

We say "stochastically increasing" rather than the previously introduced "positive regression dependent" in order to have a terminology consistent with the usual notion of stochastic ordering, which we find it convenient to employ.

Lehmann (1967), Example 1, considers the construction $S_1 = h_1(U_1,T)$, $S_2 = h_2(U_2,T)$, where U_1, U_2, T are independent and h_1, h_2 are functions increasing in T. We show:

Theorem 1.2. Let $T_1,\dots,T_n$ be associated. Let $S_i = h_i(U_i,T_1,\dots,T_n)$, $i = 1,\dots,m$, where $U_1,\dots,U_m$ are mutually independent and also independent of $T_1,\dots,T_n$ and h_i is increasing in $T_1,\dots,T_n$. Then $S_1,\dots,S_m$ are associated.

To prove Theorems 1.1 and 1.2 we consider a more general result (Theorem 3.1, or alternately Theorem 3.4) which includes both as special cases.

This investigation is primarily motivated by an implication of Theorem 1.1; random variables $T_1,\dots,T_n$ are associated if each T_i is stochastically increasing in $T_1,\dots,T_{i-1}$. This fact is useful in reliability analyses involving maintenance, spares, and queueing for repair. See Section 4 for examples. We will discuss these applications in more detail in a forthcoming document on maintenance models.

2. Representation of Stochastically Increasing Random Variables

Let S and T be random variables. Let $\underset{\sim}{S} = \{S_1,\ldots,S_n\}$ and $\underset{\sim}{T} = \{T_1,\ldots,T_n\}$ be sets of random variables. $\underset{\sim}{S}$ is *stochastically equal* to $\underset{\sim}{T}$, written $\underset{\sim}{S} =^{st} \underset{\sim}{T}$, if $\underset{\sim}{S}$ and $\underset{\sim}{T}$ have the same probability distribution. S is *stochastically less* than T, written $S \le^{st} T$, if $P[S>u] \le P[T>u]$ for all u. S is *stochastically increasing* in $\underset{\sim}{T}$, written $S \uparrow st$ in $\underset{\sim}{T}$, if $P[S>u|\underset{\sim}{T}=\underset{\sim}{t}^{(1)}] \le P[S>u|\underset{\sim}{T}=\underset{\sim}{t}^{(2)}]$ for all $\underset{\sim}{t}^{(1)} \le \underset{\sim}{t}^{(2)}$, i.e., $t_i^{(1)} \le t_i^{(2)}$, $i = 1,\ldots,n$.

Let $\underset{\sim}{S}|\underset{\sim}{T} = \underset{\sim}{t}$ denote a set of random variables with the conditional probability distribution of $\underset{\sim}{S}$, given that $\underset{\sim}{T} = \underset{\sim}{t}$.

We will use the following readily verified facts without further reference. $\underset{\sim}{S},\underset{\sim}{T} =^{st} \underset{\sim}{S}',\underset{\sim}{T}$ is equivalent to $\underset{\sim}{S}|\underset{\sim}{T} = \underset{\sim}{t} =^{st} \underset{\sim}{S}'|\underset{\sim}{T} = \underset{\sim}{t}$ for all $\underset{\sim}{t}$. $S \uparrow st$ in $\underset{\sim}{T}$ is equivalent to $S|\underset{\sim}{T} = \underset{\sim}{t}^{(1)} \le^{st} S|\underset{\sim}{T} = \underset{\sim}{t}^{(2)}$ for all $\underset{\sim}{t}^{(1)} \le \underset{\sim}{t}^{(2)}$. $f(\underset{\sim}{S},\underset{\sim}{T})|\underset{\sim}{T} = \underset{\sim}{t} =^{st} f(\underset{\sim}{S},\underset{\sim}{t})|\underset{\sim}{T} = \underset{\sim}{t}$, for any function $f(\underset{\sim}{s},\underset{\sim}{t})$.

The following lemma is a variation on a basic result due to Lehmann (1959), p. 73.

Lemma 2.1. Let $S \uparrow st$ in $\underset{\sim}{T}$. Then there exists an increasing function $h(u,\underset{\sim}{t})$, such that $S,\underset{\sim}{T} =^{st} h(U,\underset{\sim}{T}),\underset{\sim}{T}$, where U is a random variable independent of $\underset{\sim}{T}$.

Proof. Let $F_{\underset{\sim}{t}}$ be the distribution function of $S|\underset{\sim}{T} = \underset{\sim}{t}$, i.e., $F_{\underset{\sim}{t}}(s) = P[S \le s|\underset{\sim}{T}=\underset{\sim}{t}]$. Let $h(u,\underset{\sim}{t}) = \inf\{s: u \le F_{\underset{\sim}{t}}(s)\}$. h is increasing in u by its definition. $S \uparrow st$ in $\underset{\sim}{T}$ implies $h(u,\underset{\sim}{t}^{(1)}) \le h(u,\underset{\sim}{t}^{(2)})$ for $\underset{\sim}{t}^{(1)} \le \underset{\sim}{t}^{(2)}$. Thus h is increasing in $\underset{\sim}{t}$. Since $F_{\underset{\sim}{t}}$ is continuous from the right in s, $h(u,\underset{\sim}{t}) \le s \iff u \le F_{\underset{\sim}{t}}(s)$. Let U be uniformly distributed on $[0,1]$. Then $P[h(U,\underset{\sim}{t}) \le s] = P[U \le F_{\underset{\sim}{t}}(s)] = F_{\underset{\sim}{t}}(s)$, i.e., $h(U,\underset{\sim}{t}) \overset{st}{=} S|\underset{\sim}{T} = \underset{\sim}{t}$. Let U be independent of T. Then $h(U,\underset{\sim}{t})|\underset{\sim}{T} = \underset{\sim}{t} \overset{st}{=} h(U,\underset{\sim}{t})$. Thus

$$S|\underset{\sim}{T} = \underset{\sim}{t} \overset{st}{=} h(U,\underset{\sim}{t}) \overset{st}{=} h(U,\underset{\sim}{t})|\underset{\sim}{T} = \underset{\sim}{t}$$
$$\overset{st}{=} h(U,\underset{\sim}{T})|\underset{\sim}{T} = \underset{\sim}{t}.$$

It follows that $S,\underset{\sim}{T} \overset{st}{=} h(U,\underset{\sim}{T}),\underset{\sim}{T}$. ||

It is immediate that if $S \overset{st}{=} h(U,\underset{\sim}{T})$, where $h(u,\underset{\sim}{t})$ is increasing in $\underset{\sim}{t}$ and U is independent of $\underset{\sim}{T}$, then $S \uparrow$ st in $\underset{\sim}{T}$.

$S_1,\dots,S_m$ are *conditionally independent*, given that $\underset{\sim}{T} = \underset{\sim}{t}$, if $\underset{\sim}{S}|\underset{\sim}{T} = \underset{\sim}{t} \overset{st}{=} \{S_1|\underset{\sim}{T} = \underset{\sim}{t},\dots,S_m|\underset{\sim}{T} = \underset{\sim}{t}\}$, where $S_1|\underset{\sim}{T} = \underset{\sim}{t},\dots,S_m|\underset{\sim}{T} = \underset{\sim}{t}$ are assumed to be mutually independent.

Corollary 2.2. Let $S_1,\dots,S_m$ be conditionally independent, given $\underset{\sim}{T} = \underset{\sim}{t}$, for all $\underset{\sim}{t}$. Let $S_i \uparrow$ st in $\underset{\sim}{T}$, $i = 1,\dots,m$. Then there exist increasing functions $h_1,\dots,h_m$ and mutually independent random variables $U_1,\dots,U_m$ that are independent of $\underset{\sim}{T}$, such that

$$\underset{\sim}{S},\underset{\sim}{T} \overset{st}{=} \{h_1(U_1,\underset{\sim}{T}),\dots,h_m(U_m,\underset{\sim}{T})\},\ \underset{\sim}{T}.$$

Proof. Since $S_i \uparrow$ st in $\underset{\sim}{T}$, set $S_i,\underset{\sim}{T} \overset{st}{=} h_i(U_i,\underset{\sim}{T}),\underset{\sim}{T}$ in accordance with Lemma 2.1. Then $S_i|\underset{\sim}{T} = \underset{\sim}{t} \overset{st}{=} h_i(U_i,\underset{\sim}{t})$. Let $U_1,\dots,U_m$ be mutually independent. Then since $S_1,\dots,S_m$ are conditionally independent, given $\underset{\sim}{T} = \underset{\sim}{t}$,

$$\underset{\sim}{S} \mid \underset{\sim}{T} = \underset{\sim}{t} = {}^{st}\{h_1(U_1,\underset{\sim}{t}),\ldots,h_m(U_m,\underset{\sim}{t})\}$$
$$= {}^{st}\{h_1(U_1,\underset{\sim}{t}),\ldots,h_m(U_m,\underset{\sim}{t})\} \mid \underset{\sim}{T} = \underset{\sim}{t}$$
$$= {}^{st}\{h_1(U_1,\underset{\sim}{T}),\ldots,h_m(U_m,\underset{\sim}{T})\} \mid \underset{\sim}{T} = \underset{\sim}{t}.$$

Thus $\underset{\sim}{S}, \underset{\sim}{T} = {}^{st}\{h_1(U_1,\underset{\sim}{T}),\ldots,h_m(U_m,\underset{\sim}{T})$ $,\underset{\sim}{T}$. ||

<u>Theorem 2.3</u>. Let $S_1,\ldots,S_m$ be conditionally independent, given $\underset{\sim}{T} = \underset{\sim}{t}$, for all $\underset{\sim}{t}$. Let $S_i \uparrow$ st in $\underset{\sim}{T}$, $i = 1,\ldots,m$. Let $f(\underset{\sim}{s},\underset{\sim}{t})$ be an increasing function. Then $f(\underset{\sim}{S},\underset{\sim}{T}) \uparrow$ st in $\underset{\sim}{T}$.

<u>Proof</u>. Set $\underset{\sim}{S},\underset{\sim}{T} = {}^{st}\{h_1(U_1,\underset{\sim}{T}),\ldots,h_m(U_m,\underset{\sim}{T})\},\underset{\sim}{T}$ in accordance with Corollary 2.2. Let $\xi(\underset{\sim}{u},\underset{\sim}{t}) = f[h_1(u_1,\underset{\sim}{t}),\ldots,h_m(u_m,\underset{\sim}{t}),\underset{\sim}{t}]$. Then ξ is increasing, and $f(\underset{\sim}{S},\underset{\sim}{T}) = {}^{st}\xi(\underset{\sim}{U},\underset{\sim}{T})$. For $\underset{\sim}{t}^{(1)} \le \underset{\sim}{t}^{(2)}$

$$\xi(\underset{\sim}{U},\underset{\sim}{T}) \mid \underset{\sim}{T} = \underset{\sim}{t}^{(1)} = {}^{st}\xi(\underset{\sim}{U},\underset{\sim}{t}^{(1)}) \mid \underset{\sim}{T} = \underset{\sim}{t}^{(1)} = {}^{st}\xi(\underset{\sim}{U},\underset{\sim}{t}^{(1)})$$
$$\le {}^{st}\xi(\underset{\sim}{U},t^{(2)}) = {}^{st}\xi(\underset{\sim}{U},\underset{\sim}{t}^{(2)}) \mid \underset{\sim}{T} = \underset{\sim}{t}^{(2)} = {}^{st}\xi(\underset{\sim}{U},\underset{\sim}{T}) \mid \underset{\sim}{T} = \underset{\sim}{t}^{(2)}.$$

Thus $\xi(\underset{\sim}{U},\underset{\sim}{T}) \uparrow$ st in $\underset{\sim}{T}$, i.e., $f(\underset{\sim}{S},\underset{\sim}{T}) \uparrow$ st in $\underset{\sim}{T}$. ||

3. <u>Stochastically Increasing Random Variables, and Association</u>

Theorems 1.1 and 1.2 are both special cases of:

<u>Theorem 3.1</u>. Let $T_1,\ldots,T_n$ be associated. Let $S_1,\ldots,S_m$ be conditionally independent, given $\underset{\sim}{T} = \underset{\sim}{t}$, for all $\underset{\sim}{t}$. Let $S_i \uparrow$ st in $\underset{\sim}{T}$, $i = 1,\ldots,m$. Then $S_1,\ldots,S_m$, $T_1,\ldots,T_n$ are associated.

<u>Proof A</u>. Set $\underset{\sim}{S},\underset{\sim}{T} = {}^{st}\{h_1(U_1,\underset{\sim}{T}),\ldots,h_m(U_m,\underset{\sim}{T})\},\underset{\sim}{T}$ in accordance with Corollary 2.2. Since $U_1,\ldots,U_m$ are mutually independent, then $U_1,\ldots,U_m$ are associated [E-P-W(1967), Theorem 2.1]. Since $\underset{\sim}{U},\underset{\sim}{T}$ are independent,

then $U_1,\ldots,U_m$, $T_1,\ldots,T_n$ are associated [E-P-W(1967), Property P_2]. Let $f(\underset{\sim}{s},\underset{\sim}{t})$, $g(\underset{\sim}{s},\underset{\sim}{t})$ be increasing functions such that $\mathrm{Cov}[f(\underset{\sim}{S},\underset{\sim}{T}),g(\underset{\sim}{S},\underset{\sim}{T})]$ exists. Let $\xi(\underset{\sim}{u},\underset{\sim}{t}) = f[h_1(u_1,\underset{\sim}{t}),\ldots,h_m(u_m,\underset{\sim}{t}),\underset{\sim}{t}]$, $\eta(\underset{\sim}{u},\underset{\sim}{t}) = g[h_1(u_1,\underset{\sim}{t}),\ldots,h_m(u_m,\underset{\sim}{t}),\underset{\sim}{t}]$. ξ,η are increasing functions, and $f(\underset{\sim}{S},\underset{\sim}{T}),g(\underset{\sim}{S},\underset{\sim}{T}) \overset{st}{=} \xi(\underset{\sim}{U},\underset{\sim}{T}),\eta(\underset{\sim}{U},\underset{\sim}{T})$. Thus

$$\mathrm{Cov}[f(\underset{\sim}{S},\underset{\sim}{T}),g(\underset{\sim}{S},\underset{\sim}{T})] = \mathrm{Cov}[\xi(\underset{\sim}{U},\underset{\sim}{T}),\eta(\underset{\sim}{U},\underset{\sim}{T})] \geq 0,$$

and so $S_1,\ldots,S_m$, $T_1,\ldots,T_n$ are associated. ||

We find the expectation of a function $f(\underset{\sim}{S},\underset{\sim}{T})$ by first conditioning on $\underset{\sim}{T}$, i.e.,

$$Ef(\underset{\sim}{S},\underset{\sim}{T}) = E_{\underset{\sim}{T}}E_{\underset{\sim}{S}|\underset{\sim}{T}}f(\underset{\sim}{S},\underset{\sim}{T})$$

where $E_{\underset{\sim}{T}}$ denotes expectation over the distribution of $\underset{\sim}{T}$, and $E_{\underset{\sim}{S}|\underset{\sim}{T}}$ denotes expectation over the conditional distribution of $\underset{\sim}{S}$, given a fixed $\underset{\sim}{T}$.

<u>Proof B</u>. Let $f(\underset{\sim}{s},\underset{\sim}{t})$, $g(\underset{\sim}{s},\underset{\sim}{t})$ be increasing functions such that $\mathrm{Cov}[f(\underset{\sim}{S},\underset{\sim}{T}),g(\underset{\sim}{S},\underset{\sim}{T})]$ exists. Then, dropping arguments,

$$
\begin{aligned}
(3.1)\qquad \mathrm{Cov}[f,g] &= Efg - EfEg \\
&= E_{\underset{\sim}{T}}E_{\underset{\sim}{S}|\underset{\sim}{T}}fg - \{E_{\underset{\sim}{T}}E_{\underset{\sim}{S}|\underset{\sim}{T}}f\}\{E_{\underset{\sim}{T}}E_{\underset{\sim}{S}|\underset{\sim}{T}}g\} \\
&= E_{\underset{\sim}{T}}E_{\underset{\sim}{S}|\underset{\sim}{T}}fg - E_{\underset{\sim}{T}}\{E_{\underset{\sim}{S}|\underset{\sim}{T}}fE_{\underset{\sim}{S}|\underset{\sim}{T}}g\} \\
&\quad + E_{\underset{\sim}{T}}\{E_{\underset{\sim}{S}|\underset{\sim}{T}}fE_{\underset{\sim}{S}|\underset{\sim}{T}}g\} - \{E_{\underset{\sim}{T}}E_{\underset{\sim}{S}|\underset{\sim}{T}}f\}\{E_{\underset{\sim}{T}}E_{\underset{\sim}{S}|\underset{\sim}{T}}g\} \\
&= E_{\underset{\sim}{T}}\,\mathrm{Cov}_{\underset{\sim}{S}|\underset{\sim}{T}}[f,g] + \mathrm{Cov}_{\underset{\sim}{T}}[E_{\underset{\sim}{S}|\underset{\sim}{T}}f,E_{\underset{\sim}{S}|\underset{\sim}{T}}g].
\end{aligned}
$$

Let $V_1 \overset{st}{=} S_1|\underset{\sim}{T} = \underset{\sim}{t},\ldots,$ $V_m \overset{st}{=} S_m|\underset{\sim}{T} = \underset{\sim}{t}$. Since $V_1,\ldots,V_m$ are independent, $V_1,\ldots,V_m$ are associated [E-P-W(1967), Theorem 2.1].

Then $\{f(\underset{\sim}{S},\underset{\sim}{T}),g(\underset{\sim}{S},\underset{\sim}{T})\}|\underset{\sim}{T}=\underset{\sim}{t} =^{st} \{f(\underset{\sim}{S},\underset{\sim}{t}),g(\underset{\sim}{S},\underset{\sim}{t})\}|\underset{\sim}{T}=\underset{\sim}{t} =^{st} f(\underset{\sim}{V},\underset{\sim}{t}),\ g(\underset{\sim}{V},\underset{\sim}{t})$,

and

$$\mathrm{Cov}_{\underset{\sim}{S}|\underset{\sim}{T}=\underset{\sim}{t}}[f(\underset{\sim}{S},\underset{\sim}{T}),g(\underset{\sim}{S},\underset{\sim}{T})] = \mathrm{Cov}[f(\underset{\sim}{V},\underset{\sim}{t}),g(\underset{\sim}{V},\underset{\sim}{t})] \geq 0,$$

by the definition of association. Thus

$$E_{\underset{\sim}{T}}\mathrm{Cov}_{\underset{\sim}{S}|\underset{\sim}{T}}[f(\underset{\sim}{S},\underset{\sim}{T}),g(\underset{\sim}{S},\underset{\sim}{T})] \geq 0. \tag{3.2}$$

Let $\lambda(\underset{\sim}{t}) = E_{\underset{\sim}{S}|\underset{\sim}{T}=\underset{\sim}{t}}f(\underset{\sim}{S},\underset{\sim}{T})$, $\mu(\underset{\sim}{t}) = E_{\underset{\sim}{S}|\underset{\sim}{T}=\underset{\sim}{t}}g(\underset{\sim}{S},\underset{\sim}{T})$. Since $f(\underset{\sim}{S},\underset{\sim}{T}) \uparrow$ st in $\underset{\sim}{T}$, $g(\underset{\sim}{S},\underset{\sim}{T}) \uparrow$ st in $\underset{\sim}{T}$ by Theorem 2.3, then $\lambda(\underset{\sim}{t}),\mu(\underset{\sim}{t})$ are increasing functions. Since $T_1,\ldots,T_n$ are associated,

$$\mathrm{Cov}_{\underset{\sim}{T}}[E_{\underset{\sim}{S}|\underset{\sim}{T}}f(\underset{\sim}{S},\underset{\sim}{T}),E_{\underset{\sim}{S}|\underset{\sim}{T}}g(\underset{\sim}{S},\underset{\sim}{T})] = \mathrm{Cov}[\lambda(\underset{\sim}{T}),\mu(\underset{\sim}{T})] \geq 0. \tag{3.3}$$

From (3.1), (3.2), and (3.3), $\mathrm{Cov}[f(\underset{\sim}{S},\underset{\sim}{T}),g(\underset{\sim}{S},\underset{\sim}{T})] \geq 0$, so that $S_1,\ldots,S_m$, $T_1,\ldots,T_n$ are associated. ||

The following multivariate definitions of "stochastically less than" and "stochastically increasing" are of interest in the present context, and also because of their apparent relevance to reliability theory:

Definition 3.2. $\underset{\sim}{S}$ is *stochastically less* than $\underset{\sim}{S}'$, written $\underset{\sim}{S} \leq^{st} \underset{\sim}{S}'$, if $f(\underset{\sim}{S}) \leq^{st} f(\underset{\sim}{S}')$ for all increasing functions $f(\underset{\sim}{s})$.

Definition 3.3. $\underset{\sim}{S}$ is *stochastically increasing* in $\underset{\sim}{T}$, written $\underset{\sim}{S} \uparrow$ st in $\underset{\sim}{T}$, if $f(\underset{\sim}{S}) \uparrow$ st in $\underset{\sim}{T}$ for all increasing functions $f(\underset{\sim}{s})$.

It is immediate that $\underset{\sim}{S} \uparrow$ st in $\underset{\sim}{T}$ is equivalent to $\underset{\sim}{S}|\underset{\sim}{T} = \underset{\sim}{t}^{(1)} \leq^{st} \underset{\sim}{S}|\underset{\sim}{T} = \underset{\sim}{t}^{(2)}$ for all $\underset{\sim}{t}^{(1)} \leq \underset{\sim}{t}^{(2)}$. From Theorem 2.3, if

$S_1,\dots,S_m$ are conditionally independent, given $\underset{\sim}{T} = \underset{\sim}{t}$, for all $\underset{\sim}{t}$, and $S_i \uparrow$ st in $\underset{\sim}{T}$, $i = 1,\dots,m$, then $\underset{\sim}{S} \uparrow$ st in $\underset{\sim}{T}$, where $\underset{\sim}{S} = (S_1,\dots,S_m)$.

Lemma 3.4. Let $f(\underset{\sim}{s},\underset{\sim}{t})$ be an increasing function. Let $\underset{\sim}{S} \uparrow$ st in $\underset{\sim}{T}$. Then $f(\underset{\sim}{S},\underset{\sim}{T}) \uparrow$ st in $\underset{\sim}{T}$.

Proof. For $\underset{\sim}{t}^{(1)} \le \underset{\sim}{t}^{(2)}$,

$$f(\underset{\sim}{S},\underset{\sim}{T})\,|\,\underset{\sim}{T} = \underset{\sim}{t}^{(1)} \overset{st}{=} f(\underset{\sim}{S},\underset{\sim}{t}^{(1)})\,|\,\underset{\sim}{T} = \underset{\sim}{t}^{(1)} \overset{st}{\le} f(\underset{\sim}{S},\underset{\sim}{t}^{(2)})\,|\,\underset{\sim}{T} = \underset{\sim}{t}^{(1)}$$

$$\overset{st}{\le} f(\underset{\sim}{S},\underset{\sim}{t}^{(2)})\,|\,\underset{\sim}{T} = \underset{\sim}{t}^{(2)} \overset{st}{=} f(\underset{\sim}{S},\underset{\sim}{T})\,|\,\underset{\sim}{T} = \underset{\sim}{t}^{(2)}.$$

Thus $f(\underset{\sim}{S},\underset{\sim}{T})\,|\,\underset{\sim}{T} = \underset{\sim}{t}^{(1)} \overset{st}{\le} f(\underset{\sim}{S},\underset{\sim}{T})\,|\,\underset{\sim}{T} = \underset{\sim}{t}^{(2)}$ for all $\underset{\sim}{t}^{(1)} \le \underset{\sim}{t}^{(2)}$, i.e., $f(\underset{\sim}{S},\underset{\sim}{T}) \uparrow$ st in $\underset{\sim}{T}$. ||

$S_1,\dots,S_m$ are *conditionally associated*, given $\underset{\sim}{T} = \underset{\sim}{t}$, if $\{S_1,\dots,S_m \mid \underset{\sim}{T} = \underset{\sim}{t}\}$ are associated.

Theorem 3.1 is a special case of:

Theorem 3.5. Let $T_1,\dots,T_n$ be associated. Let $S_1,\dots,S_m$ be conditionally associated, given $\underset{\sim}{T} = \underset{\sim}{t}$, for all $\underset{\sim}{t}$. Let $\underset{\sim}{S} \uparrow$ st in $\underset{\sim}{T}$. Then $S_1,\dots,S_m, T_1,\dots,T_n$ are associated.

The proof follows the lines of Proof B of Theorem 3.1, using Lemma 3.4 in place of Theorem 2.3.

4. Applications and Examples

Random variables $T_1,\dots,T_n$ are stochastically [illegible] if $T_2 \uparrow$ st in T_1, $T_3 \uparrow$ st in $\{T_1, T_2\}, \ldots, T_n \uparrow$ st in $\{T_1, \ldots, T_{n-1}\}$.

<u>Theorem 4.1</u>. Let $T_1, \ldots, T_n$ be stochastically increasing in sequence. Then $T_1, \ldots, T_n$ are associated.

<u>Proof</u>. $\{T_1\}$ is a set of associated random variables [E-P-W(1967), Property P_3]. Since $T_2 \uparrow$ st in T_1, T_1, T_2 are associated by Theorem 1.1. Continuing by induction, using Theorem 1.1, $T_1, \ldots, T_n$ are associated. ||

A random variable T has a *decreasing failure rate* (DFR) distribution [Barlow-Marshall-Proschan(1963)] if $\overline{F}(t+u)/\overline{F}(t)$ is increasing in t for all $u \geq 0$, where $\overline{F}(t) = P[T>u]$.

<u>Example 4.2</u>. Let $T^{(1)} \leq \ldots \leq T^{(n)}$ be the order statistics in a sample of size n from a DFR distribution. Let $D_1 = T_1$ and $D_i = T^{(i)} - T^{(i-1)}$, $i = 2, \ldots, n$. Then $D_1, \ldots, D_n$ are stochastically increasing in sequence.

<u>Proof</u>. Note that

$$P[D_{i+1} > u \mid D_1 = d_1, \ldots, D_i = d_i] = \left\{ \frac{\overline{F}(d_1 + \ldots + d_i + u)}{\overline{F}(d_1 + \ldots + d_i)} \right\}^{n-i}$$

is increasing in $d_1, \ldots, d_i$ for all $u \geq 0$. Thus $D_{i+1} \uparrow$ st in $D_1, \ldots, D_i$, and so $D_1, \ldots, D_n$ are stochastically increasing in sequence. ||

It follows from Theorem 4.1 that $D_1, \ldots, D_n$ are associated.

A stochastic process $\{X(t), t \in \tau\}$ is *associated in time* if the random variables $X(t_1), \ldots, X(t_k)$ are associated for all k and all $\{t_1, \ldots, t_k\} \subset \tau$. $\{X(t), t \in \tau\}$ is *stochastically increasing in time* if $X(t_1), \ldots, X(t_k)$ are stochastically increasing in sequence for all k and all $\{t_1 < \dots < t_k\} \subset \tau$. It is equivalent to say that $\{X(t), t \in \tau\}$ is stochastically increasing in time if $X(t_k) \uparrow$ st in $\{X(t_1), \dots, X(t_{k-1})\}$ for all $\{t_1 < \dots < t_k\} \subset \tau$.

Theorem 4.3. Let $\{X(t), t \in \tau\}$ be stochastically increasing in time. Then $\{X(t), t \in \tau\}$ is associated in time.

Proof. For any k and $\{t_1 < \dots < t_k\} \subset \tau$, $X(t_1), \dots, X(t_k)$ are stochastically increasing in sequence. By Theorem 4.1 $X(t_1), \dots, X(t_k)$ are associated. Thus $\{X(t), t \in \tau\}$ is associated in time.||

Theorem 4.4. Let $\{X(t), t \in \tau\}$ be a Markov process. Let $X(t_2) \uparrow$ st in $X(t_1)$ for all $\{t_1 < t_2\} \subset \tau$. Then $\{X(t), t \in \tau\}$ is stochastically increasing in time.

Proof. Let $\{t_1 < \dots < t_k\} \subset \tau$. Since $\{X(t), t \in \tau\}$ is a Markov process, $X(t_k) | \{X(t_1) = x_1, \dots, X(t_{k-1}) = x_{k-1}\} =^{st} X(t_k) | X(t_{k-1}) = x_{k-1}$, all $x_1, \dots, x_{k-1}$. Let $x_1 \leq y_1, \dots, x_{k-1} \leq y_{k-1}$. Since $X(t_k) \uparrow$ st in $X(t_{k-1})$, $X(t_k) | X(t_{k-1}) = x_{k-1} \leq^{st} X(t_k) | X(t_{k-1}) = y_{k-1}$. Then

$$X(t_k) | \{X(t_1) = x_1, \dots, X(t_{k-1}) = x_{k-1}\} =^{st} X(t_k) | X(t_{k-1}) = x_{k-1}$$
$$\leq^{st} X(t_k) | X(t_{k-1}) = y_{k-1} =^{st} X(t_k) | \{X(t_1) = y_1, \dots, X(t_{k-1}) = y_{k-1}\}.$$

Thus $X(t_k) \uparrow$ st in $\{X(t_1), \dots, X(t_{k-1})\}$, i.e., $\{X(t), t \in \tau\}$ is stochastically increasing in time. ||

Example 4.5. Let $\{X(t), t \in \tau\}$ be a Markov process such that $X(t) = 0$ or 1 for each $t \in \tau$. Let

$$P[X(t_2) = 1 \mid X(t_1) = 1] = p(t_1,t_2)$$
$$P[X(t_2) = 0 \mid X(t_1) = 0] = q(t_1,t_2)$$

where $p(t_1,t_2) + q(t_1,t_2) \geq 1$ for all $\{t_1 < t_2\} \subset \tau$. Then $\{X(t),\ t \in \tau\}$ is stochastically increasing in time.

Proof. For $\{t_1 < t_2\} \subset \tau$, set $U(t_1,t_2) =^{st} X(t_2) | X(t_1) = 1$, $V(t_1,t_2) =^{st} X(t_2) | X(t_1) = 0$. Then $P[V(t_1,t_2) = 1] = 1 - q(t_1,t_2) \leq p(t_1,t_2) = P[U(t_1,t_2) = 1]$, i.e., $V(t_1,t_2) \leq^{st} U(t_1,t_2)$. Thus $X(t_2) \uparrow st$ in $X(t_1)$, and by Theorem 4.4 $\{X(t),\ t \in \tau\}$ is stochastically increasing in time. ||

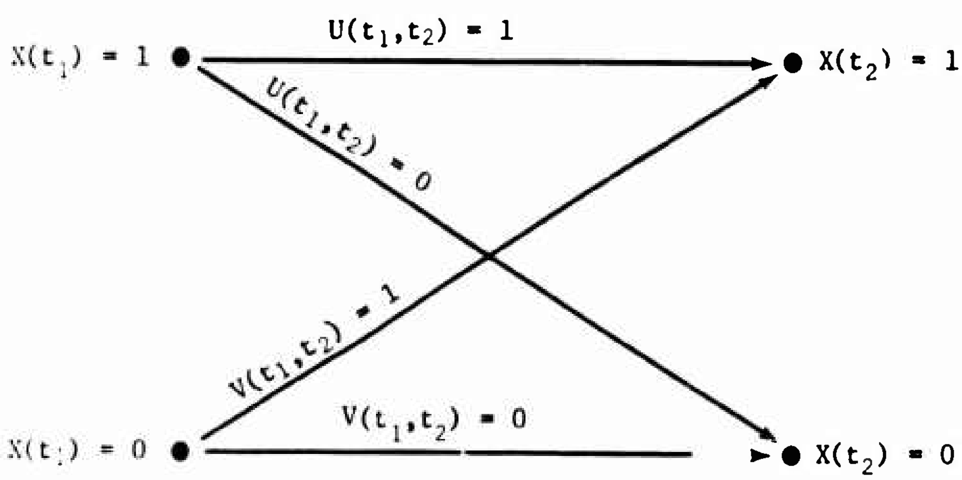
In reliability theory, processes of the type considered in Example 4.5 are basic models for the performance of a device subject to alternate failure and repair, where $X(t) = 1$ if the device is functioning at time t, $X(t) = 0$ if the device is failed at time t. If $\tau = \{0,1,\dots\}$ and $p(k,k+1) = p$, $q(k,k+1) = q$, then $\{X(t),\ t \in \tau\}$ corresponds to an alternating renewal process where time from repair to failure has a geometric distribution with parameter p, and time from failure to repair has a geometric distribution with parameter q. This geometric-geometric performance process is stochastically increasing in time if $p + q \geq 1$. If $\tau = [0,+\infty)$ and

$$p(t_1,t_2) = (\lambda+\mu)^{-1}\{\mu + \lambda \exp[-(\lambda+\mu)(t_2-t_1)]\}$$

$$q(t_1,t_2) = (\lambda+\mu)^{-1}\{\lambda + \mu \exp[-(\lambda+\mu)(t_2-t_1)]\},$$

then $\{X(t),\ t \in \tau\}$ corresponds to an alternating renewal process where time from repair to failure has an exponential distribution with parameter λ, and time from failure to repair has an exponential distribution with parameter μ. Since $p(t_1,t_2) + q(t_1,t_2) \geq 1$ for all $t_1 < t_2$, $\lambda \geq 0$, $\mu \geq 0$, the exponential-exponential performance process is stochastically increasing in time. It follows from Theorem 4.3 that the exponential-exponential performance process and the geometric-geometric performance process with $p + q \geq 1$ are associated in time [cf. E-P-W(1966)]. Some properties and applications in reliability theory of performance processes that are associated in time are discussed in E-P-W(1966).

In the process of Example 4.5 $\{X(t_1), X(t_2)\}$ are, for $\{t_1 < t_2\} \subset \tau$, stochastically representable by $X(t_1)$ and transition random variables $U(t_1,t_2) \overset{st}{=} X(t_2)|X(t_1) = 1$, $V(t_1,t_2) \overset{st}{=} X(t_2)|X(t_1) = 0$, such that $X(t_1)$, $U(t_1,t_2)$, $V(t_1,t_2)$ are mutually independent.



$\{X(t_1), X(t_2)\} \overset{st}{=} \{X(t_1), X(t_1)U(t_1,t_2) + [1-X(t_1)]V(t_1,t_2)\}$, and so $X(t_2)$ ↑ st in $X(t_1)$ is equivalent to $V(t_1,t_2) \leq^{st} U(t_1,t_2)$. Setting $U(t_1,t_2)$, $V(t_1,t_2)$ independent is convenient, but not essential to the representation.

In a frequently studied reliability model involving a complex of n identical devices, the functioning devices are in various degrees of service or standby for service, and the failed devices are in various degrees of repair or standby for repair. In this model the basic descriptor of performance is $X(t)$ = the number of devices functioning at time t. The following generalization of Example 4.5 covers a variety of cases in which the process $\{X(t), t \in \tau\}$ is stochastically increasing in time, and thus associated in time.

Example 4.6. Let $\{X(t), t \in \tau\}$ be a Markov process such that $X(t) = 0$ or 1 or ... or n for each $t \in \tau$. For all $\{t_1 < t_2\} \subset \tau$ let

$$X(t_2)|X(t_1) = i \stackrel{st}{=} U_1(t_1,t_2) + \dots + U_i(t_1,t_2) + V_{i+1}(t_1,t_2) + \dots + V_n(t_1,t_2),$$

$i = 0,\dots,n$, where $U_i(t_1,t_2) = 0$ or 1, $V_i(t_1,t_2) = 0$ or 1, $i = 1,\dots,n$, $\{U_1(t_1,t_2), V_1(t_1,t_2)\}, \dots, \{U_n(t_1,t_2), V_n(t_1,t_2)\}$ are mutually independent couples, and $V_i(t_1,t_2) \stackrel{st}{\le} U_i(t_1,t_2)$, $i = 1,\dots,n$. Then $\{X(t), t \in \tau\}$ is stochastically increasing in time.

Proof. Set

$$Z_j(t_2)|X(t_1) = i \stackrel{st}{=} \begin{cases} U_j(t_1,t_2) & \text{if } i \ge j \\ V_j(t_1,t_2) & \text{if } i < j, \end{cases}$$

$j = 1,\dots,n$. Since $V_j(t_1,t_2) \stackrel{st}{\le} U_j(t_1,t_2)$, $Z_j(t_2) \uparrow st$ in $X(t_1)$. Let $Z_1(t_2),\dots,Z_n(t_2)$ be conditionally independent, given $X(t_1) = i$, $i = 0,\dots,n$. Then

$$X(t_2) \mid X(t_1) = i \overset{st}{=} \sum_{j=1}^{n} \{Z_j(t_2) \mid X(t_1) = i\} \overset{st}{=} \left\{\sum_{j=1}^{n} Z_j(t_2)\right\} \mid X(t_1) = i,$$

$i = 0,\ldots,n$. Thus $\{X(t_2), X(t_1)\} \overset{st}{=} \left\{\sum_{j=1}^{n} Z_j(t_2), X(t_1)\right\}$. Since $\sum_{j=1}^{n} Z_j(t_2) \uparrow st$ in $X(t_1)$ by Theorem 2.3, $X(t_2) \uparrow st$ in $X(t_1)$. ||

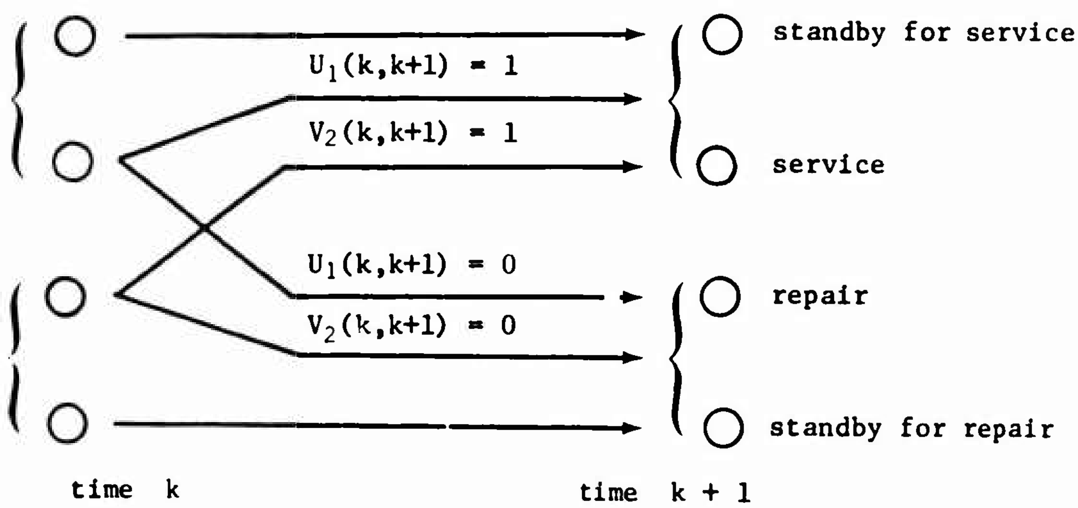
We illustrate the application of Example 4.6 for plans involving two identical devices, $n = 2$, where time is measured in discrete cycles, say $T = \{0,1,\ldots\}$. We suppose that devices fail or are repaired within cycles, and that devices are transferred from standby for service to service, service to standby for repair, etc., between cycles.

devices functioning	②	filled second
at end of k^{th} cycle	①	filled first
	service "sockets"	
devices failed	②	filled first
at end of k^{th} cycle	①	filled second
	repair "sockets"	

time k

We view the experience of a device within each cycle as independent of its experience in preceding cycles, and dependent only on the type of service or repair it is subject to on that cycle.

Case 4.6(a).



Assuming $X(t_k)$, $U_1(k,k+1)$, $V_2(k,k+1)$ mutually independent:

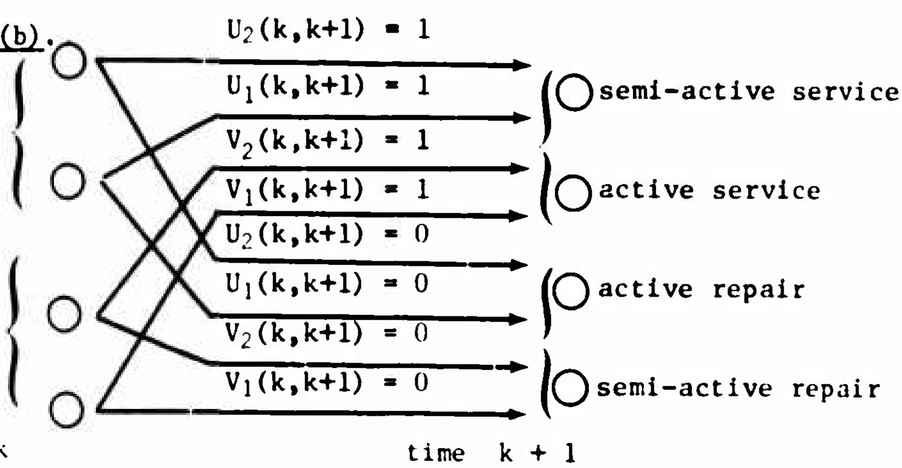
$$X(k+1)\,|\,X(k) = 2 =^{st} U_1(k,k+1) + 1.$$

$$X(k+1)\,|\,X(k) = 1 =^{st} U_1(k,k+1) + V_2(k,k+1).$$

$$X(k+1)\,|\,X(k) = 0 =^{st} 0 \qquad + V_2(k,k+1).$$

$X(k+1) \uparrow st$ in $X(k)$ by Example 4.6. ||

Case 4.6(b).



Assuming $X(k)$, $U_1(k,k+1)$, $U_2(k,k+1)$, $V_1(k,k+1)$, $V_2(k,k+1)$ mutually independent:

$$X(k+1)\,|\,X(k) = 2 =^{st} U_1(k,k+1) + U_2(k,k+1).$$
$$X(k+1)\,|\,X(k) = 1 =^{st} U_1(k,k+1) + V_2(k,k+1).$$
$$X(k+1)\,|\,X(k) = 0 =^{st} V_1(k,k+1) + V_2(k,k+1).$$

By Example 4.6, $X(k+1) \uparrow st$ in $X(k)$ if $V_1(k,k+1) \leq^{st} U_1(k,k+1)$ and $V_2(k,k+1) \leq^{st} U_2(k,k+1)$, e.g., if $U_1(k,k+1) \leq^{st} U_2(k,k+1)$, $V_1(k,k+1) \leq^{st} V_2(k,k+1)$, and $V_2(k,k+1) \leq^{st} U_1(k,k+1)$. ||

Further applications of Example 4.6, e.g., to cases in which time is measured continuously, will be considered in a forthcoming document.

REFERENCES

R. E. Barlow, A. W. Marshall, F. Proschan (1963). Properties of probability distributions with monotone hazard rate. *Ann. Math. Statist.*, 34, 375-389.

J. D. Esary, F. Proschan, D. W. Walkup (1966). A multivariate notion of association, with a reliability application. Boeing Scientific Research Laboratories Document D1-82-0567.

J. D. Esary, F. Proschan, D. W. Walkup (1967). A multivariate notion of association, with applications. *Ann. Math. Statist.*, 38, 1466-1474.

J. D. Esary and F. Proschan (1967). Relationships among some notions of bivariate dependence. Boeing Scientific Research Laboratories Document D1-82-0597.

E. L. Lehmann (1959). *Testing Statistical Hypotheses*. Wiley, New York.

E. L. Lehmann (1966). Some concepts of dependence. *Ann. Math. Statist.*, 37, 1137-1153.

J. W. Tukey (1958). A problem of Berkson, and minimum variance orderly estimators. *Ann. Math. Statist.*, 29, 588-592.